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Frequency moments, L_q norms and Rényi entropies of general hypergeometric polynomials

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Abstract The basic variables of the information theory of quantum systems (e.g., frequency or entropic moments, Rényi and Tsallis entropies) can be expressed in terms of L_q norms of general hypergeometrical polynomials. These polynomials are known to control the radial and angular parts of the wavefunctions of the quantum-mechanically allowed states of numerous physical and chemical systems. The computation of the L_q norms of these polynomials is presently an interesting issue *per se* in the theory of special functions; moreover, these quantities are closely related to the frequency moments and other information-theoretic properties of the associated Rakhmanov probability density. In this paper we calculate the unweighted and weighted L_q-norms ($q = 2k$, $k \in \mathbb{N}$) of general hypergeometric real orthogonal polynomials (Hermite, Laguerre and Jacobi) and some entropy-like integrals of Bessel polynomials, in terms of q and the parameters of the corresponding weight function by using their explicit expression and second order differential equation. In addition, the asymptotics ($q \to \infty$) of the unweighted L_q norms of the Jacobi polynomials is determined by the Laplace method.

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1 Introduction

Let us consider the polynomials $y_n(x)$ of hypergeometric type, which are solutions of

$$\sigma(x)y_n'' + \tau(x)y_n' + \lambda_n y_n = 0, \tag{1}$$

with $\lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma''$, and $\sigma(x)$ and $\tau(x)$ are polynomials of degrees, at most, 2 and 1, respectively. Moreover, $\omega(x)$ $(x \in \Delta)$ denotes the symmetrization function which satisfies the Pearson differential equation

$$[\sigma(x)\omega(x)]' = \tau(x)\omega(x), \tag{2}$$

so that Eq. (1) can be written in the self-adjoint form $(\sigma\omega y_n')' + \lambda_n\omega y_n = 0$. See [1] for further mathematical details and physical applications.

Throughout this paper, we set

$$N_q(n) \equiv \int_\Delta \omega(x)|y_n(x)|^q\,dx,\ \ q \in \mathbb{R}_+, \tag{3}$$

and

$$W_q(n) \equiv \int_\Delta |\omega(x)y_n^2(x)|^q\,dx,\ \ q \in \mathbb{R}_+, \tag{4}$$

for the unweighted and weighted L_q norms of the polynomials $y_n(x)$. The L_q norms of hypergeometric polynomials have been recently shown to be closely connected with some combinatorial objects [2] and to have some interesting information-theoretic characterizations [3–5]. It is worth pointing out here that these quantities are closely connected to several information-theoretic properties of the Rakhmanov probability density associated to the polynomial $y_n(x)$, which is defined by $\rho_n(x) = \omega(x)y_n^2(x)$. Indeed, the quantities

$$R_q[\rho_n] = \frac{1}{1-q}\ln W_q(n);\ \ q > 0, q \neq 1,$$

and

$$T_q[\rho_n] = \frac{1}{q-1}\left(1 - W_q(n)\right);\ \ q > 0, q \neq 1,$$

describe the Rényi entropy [6,7] and the Tsallis entropy [8,9] of $\rho_n(x)$, respectively.

The R_q and T_q quantities, which include the celebrated Shannon information entropy $S[\rho_n] = -\int \rho_n(x) \ln \rho_n(x)\, dx$ in the limiting case $q \to 1$, grasp different aspects of the distribution of the probability density $\rho_n(x)$ along the interval Δ when the order q is varying. Moreover, the L_q norms have been used to define various spreading measures of the polynomials $y_n(x)$ over the support interval Δ; in particular the Rényi lengths $L_q^R[\rho_n]$ for the Hermite [3], Laguerre [5] and Jacobi [4] cases, which allow us to know how these polynomials are effectively distributed on the orthogonality interval in a quantitative manner.

Physically, the Rakhmanov density $\rho_n(x)$ describes the probability density of the ground and excited states of the physical systems whose non-relativistic wavefunctions are controlled by the polynomials $y_n(x)$. The frequency moments of this density, which are given by the corresponding weighted L_q norms, allow us to gain insight into the internal disorder of the quantum systems and, moreover, they represent various fundamental and/or experimentally measurable quantities of the systems; e.g., the frequency moments of order $p = 1, 2, 4/3, 5/3$ are, at times up to a proportionality factor, the number of constituents, the average electron density, and the Dirac exchange and Thomas-Fermi energies (see e.g. [10,11]), respectively. In addition, they can be represented by functionals of the single-particle density of the physical systems [1,12,13]. For further details, see e.g. the recent review [11].

Since the times of Bernstein [14] and Steklov [15] it is known that the L_q norm of measurable functions is known to be a very useful concept in various mathematical fields ranging from classical analysis to applied mathematics and quantum physics. However, by the end of 1990's the only theoretical knowledge to calculate the L_q norms of special functions was the asymptotics ($n \to \infty$) results of the weighted L_q norms of Aptekarev et al [16,17] for the hypergeometric polynomials $y_n(x)$, and some elegant inequalities for L_q norms of specific orthogonal polynomials [18,19]. Recently, two analytical procedures to calculate weighted L_q norms for orthogonal polynomials with arbitrary degree n have been proposed [3–5]. One uses the expansion of $|y_n(x)|^q$ in terms of the powers of the variable, being the expansion coefficients given by some multivariate Bell polynomials. The other method linearizes $|y_n(x)|^q$ in terms of $y_k(x)$, being the linearization coefficients given by some multivariate special functions of Lauricella and Srivastava types [20,21].

In this paper we extend these works in a threefold sense. First, we calculate not only the weighted norms but also the unweighted ones. Second, we compute these two norms not only for the classical orthogonal polynomials in a real variable but also for the larger family of the hypergeometric polynomials. In turn, we calculate the unweighted and weighted L_q norms of the hypergeometric polynomial of degree n in terms of the expansion coefficients c_k ($k = 0, 1, \ldots, n$) of its explicit expression and the polynomial coefficients $\sigma(x)$ and $\tau(x)$ of the second order differential equation that it satisfies. Third, we determine the asymptotics ($q \to \infty$) of the unweighted L_q norms of the general orthogonal polynomials by means of the Laplace method [22], following a similar procedure recently used for the corresponding weighted norms [23]. This method works for Jacobi polynomials but it does not apply in Hermite, Laguerre and Bessel cases for reasons discussed later.

The structure of this paper is the following. In Sects. 2 and 3 we calculate the unweighted and weighted L_q norms for $q = 2k, k \in \mathbb{N}$ and $q \in \mathbb{N}$, respectively, of the three canonical families of real hypergeometric orthogonal polynomials; namely, the Hermite, Laguerre and Jacobi polynomials. In Sect. 4 we obtain the asymptotics ($q \to \infty$) of the unweighted L_q norms of the Jacobi orthogonal polynomials using the Laplace asymptotic method [22]. Then, in Sect. 5 we compute some entropy-like integrals for the Bessel polynomials [24–27].

2 L_q norms of general hypergeometric polynomials

In this section we first determine the (unweighted) L_q norms, $N_q(n)$ with $q = 2k, k \in \mathbb{N}$, of general hypergeometric polynomials $y_n(x)$. Then, we apply them to the three canonical families of real continuous orthogonal hypergeometric polynomials: Hermite, Laguerre and Jacobi. Let us assume that the polynomials $y_n(x)$ have the explicit expression

$$y_n(x) = \sum_{k=0}^{n} c_k x^k \tag{5}$$

and fulfill the second-order hypergeometric differential equation (1).

To calculate the quantities $N_q(n)$ defined by (3) for $q = 2k,\ k \in \mathbb{N}$, we begin with the following power expansion [3] (see also [28])

$$[y_n(x)]^q = \left[\sum_{k=0}^{n} c_k x^k\right]^q = \sum_{t=0}^{nq} \frac{q!}{(t+q)!} B_{t+q,q}(c_0, 2!c_1, \ldots, (t+1)!c_t) x^t \tag{6}$$

with $c_i = 0$ for $i > n$, and where the B-symbols denote the Bell polynomials of combinatorics [29], which are given by

$$B_{m,l}(c_1, c_2, \ldots, c_{m-l+1}) = \sum_{\pi(m,l)} \frac{m!}{j_1! j_2! \ldots j_{m-l+1}!} \left(\frac{c_1}{1!}\right)^{j_1} \left(\frac{c_2}{2!}\right)^{j_2} \\ \ldots \left(\frac{c_{m-l+1}}{(m-l+1)!}\right)^{j_{m-l+1}} \tag{7}$$

where the sum runs over all partitions $\pi(m, l)$ such that

$$j_1 + j_2 + \cdots + j_{m-l+1} = l, \tag{8}$$

and

$$j_1 + 2j_2 + \cdots + (m-l+1) j_{m-l+1} = m \tag{9}$$

Then, taking (6) in (3) we obtain that

$$N_q(n) = \sum_{t=0}^{nq} \frac{q!}{(t+q)!} B_{t+q,q}\,(c_0, 2!c_1, \ldots, (t+1)!c_t)\, \mu_t \tag{10}$$

where μ_t denotes the moment of order t of the symmetrization function $\omega(x)$; i.e.

$$\mu_t = \int\limits_{\Delta} x^t \omega(x)\, dx; t = 0, 1, \ldots \tag{11}$$

To calculate these moments, we multiply the Pearson equation (2) by x^t and we integrate over the support interval $\Delta \equiv (a, b)$ of $\omega(x)$; so that,

$$\int\limits_a^b x^t (\sigma\omega)'\, dx = \int\limits_a^b x^t \tau\omega\, dx$$

Integrating by parts we have

$$\int\limits_a^b \left[x^t \tau(x) + t x^{t-1}\sigma(x)\right] \omega(x)\, dx - A(t, a, b) = 0,$$

with

$$A(t, a, b) = x^t \sigma(x)\omega(x)|_a^b = b^t \sigma(b)\omega(b) - a^t \sigma(a)\omega(a) \tag{12}$$

From this expression it is straightforward to obtain the following recurrence relation

$$\left(\frac{t}{2}\sigma'' + \tau'\right)\mu_{t+1} + \left[t\sigma'(0) + \tau(0)\right]\mu_t + t\sigma(0)\mu_{t-1} - A(t, a, b) = 0 \tag{13}$$

which allows us to determine the wanted moments μ_t, with the initial conditions $\mu_0 = \int_\Delta \omega(x)\, dx$ and $\mu_{-1} = 0$.

In turn, Eqs. (10), (11) and (13) allow us to determine the L_q norms $N_q(x)$ of the polynomials of hypergeometric type $y_n(x)$ in terms of the expansion coefficients c_k $(k = 0, 1, \ldots, n)$ and the coefficients $\sigma(x)$ and $\tau(x)$ of the corresponding differential equation (1). Moreover, when these polynomials are orthogonal with respect to $\omega(x)$ on the interval Δ so that

$$\int\limits_a^b y_n(x) y_m(x)\omega(x)\, dx = d_n^2 \delta_{m,n} \tag{14}$$

(where d_n^2 is the normalization constant), it happens that $A(t, a, b) = 0$. The classical orthogonal polynomials of Hermite, Laguerre and Jacobi satisfy this condition (see e.g. [1] and main data in Table 1).

Thus, the unweighted L_q norms, $N_q(n)$, of these orthogonal polynomials are determined by Eqs. (10) and (13), where the expansion coefficients c_k are well-known in

Table 1 Classical orthogonal polynomials of Hermite, Laguerre, Jacobi and Bessel types

	Hermite	Laguerre ($\alpha > -1$)	Jacobi ($\alpha > -1, \beta > -1$)	Bessel ($\alpha = 1, 2, 3, \ldots$)
Δ	$(-\infty, +\infty)$	$(0, +\infty)$	$(-1, +1)$	Unit circle
$\omega(z)$	e^{-z^2}	$z^\alpha e^{-z}$	$(1-z)^\alpha (1+z)^\beta$	$z^\alpha e^{-2/z}$
$\sigma(z)$	1	z	$1 - z^2$	z^2
$\tau(z)$	$-2z$	$1 + \alpha - z$	$-(\alpha + \beta + 2)z + \beta - \alpha$	$(\alpha + 2)z + 2$
λ_n	$2n$	n	$n(n + \alpha + \beta + 1)$	$-n(n + \alpha + 1)$
$y_n(z)$	$H_n(z)$	$L_n^{(\alpha)}(z)$	$P_n^{(\alpha,\beta)}(z)$	$\mathcal{B}_n^{(\alpha)}(z)$

the literature (see e.g. [1]) the moments μ_t can be obtained by the following recursion relation

$$\left(\frac{t}{2}\sigma'' + \tau'\right)\mu_{t+1} + \left[t\sigma'(0) + \tau(0)\right]\mu_t + t\sigma(0)\mu_{t-1} = 0 \tag{15}$$

This expression was previously obtained by a similar method [30]. Taking into account the polynomial coefficients $\sigma(x)$ and $\tau(x)$ given in Table 1 for each of the classical orthogonal polynomials, the recursion relation (15) boils down as follows:

$$\begin{aligned} -2\mu_{t+1} + t\mu_{t-1} &= 0 \quad \text{for} \quad H_n(x) \\ -\mu_{t+1} + (t + \alpha + 1)\mu_t &= 0 \quad \text{for} \quad L_n^{(\alpha)}(x) \\ -(t + \alpha + \beta + 2)\mu_{t+1} + (\beta - \alpha)\mu_t + t\mu_{t-1} &= 0 \quad \text{for} \quad P_n^{(\alpha,\beta)}(x) \end{aligned}$$

These recurrence relations can be solved, giving rise to the following explicit expressions

$$\mu_{2t+1}(H) = 0, \;\; \mu_{2t}(H) = \Gamma\left(t + \frac{1}{2}\right) \tag{16}$$

$$\mu_t(L) = \Gamma(1 + \alpha + t) \tag{17}$$

$$\begin{aligned} \mu_t(P) = \Gamma(1+t)\Big[&(-1)^t \frac{\Gamma(1+\beta)}{\Gamma(2+t+\beta)}\, {}_2F_1\left(-\alpha, t+1; 2+t+\beta; -1\right) \\ &+ \frac{\Gamma(1+\alpha)}{\Gamma(2+t+\alpha)}\, {}_2F_1\left(-\beta, t+1; 2+t+\alpha; -1\right)\Big] \end{aligned} \tag{18}$$

for the moments μ_t, $t = 0, 1, 2, \ldots$ of the Hermite, Laguerre and Jacobi polynomials. Expressions (16) and (17) can be easily obtained from the recurrence relation (15). This is not so in the Jacobi case, where it is much easier to use definition (11) to obtain (18). Let us emphasize, however, that the three term recurrence relation (15) is very convenient for numerical purposes.

Summarizing, the expressions (10) and (16)–(18) allow one to compute the unweighted L_q norms ($q = 2k, k \in \mathbb{N}$) of the (Rakhmanov probability density associated to the) three classical families of orthogonal hypergeometric polynomials in terms of their expansion coefficients, whose values are well-known in the literature (see e.g. [1,2,31]).

3 Weighted L_q norms of general hypergeometric polynomials

In this section we determine the weighted L_q norms $W_q(n)$, $q \in \mathbb{N}$, of the general hypergeometric polynomials $y_n(x)$. Then, we apply them to the three classical families of real orthogonal hypergeometric polynomials.

According to the definition (4) and making use of expansion formula (6) of $[y_n(x)]^q$, we obtain that the weighted L_q norms $W_q(n)$ for $q \in \mathbb{N}$ can be expressed as

$$\begin{aligned} W_q(n) &= \int_\Delta [\omega(x)]^q \, [y_n(x)]^{2q} \, dx \\ &= \sum_{t=0}^{2nq} \frac{(2q)!}{(t+2q)!} B_{t+2q,2q}\left(c_0, 2!c_1, \ldots, (t+1)!c_t\right) \cdot \Omega_q(t) \end{aligned} \tag{19}$$

with the expansion coefficients c_k [see (5)] and the Ω-functional

$$\Omega_q(t) \equiv \int_\Delta x^t [\omega(x)]^q \, dx, \tag{20}$$

where $\omega(x)$ is the symmetrization function of the polynomials as defined by means of the Pearson equation and $\Delta \equiv (a, b)$. These generalized moments $\Omega_q(t)$ can be determined recurrently in terms of the coefficients $\tau(x)$ and $\sigma(x)$ of (1) by means of a procedure similar to that already used for the moments μ_t in the previous section. Indeed, integrating (20) by parts we have that

$$\int_a^b x^t [\omega(x)]^q \, dx = \frac{x^{t+1}}{t+1} [\omega(x)]^q \Big|_a^b - \int_a^b x^t \cdot x \frac{1}{t+1} q \cdot \omega'(x) \, [\omega(x)]^{q-1} \, dx \tag{21}$$

Now we multiply the Pearson equation by $[\omega(x)]^{q-1}$,

$$(\sigma(x)\omega(x))' = \tau(x)\omega(x) \longrightarrow \sigma'(x) \, [\omega(x)]^q + \sigma(x)\omega'(x) \, [\omega(x)]^{q-1} = \tau(x) \, [\omega(x)]^q$$

Taking the last term of the left side alone, i.e.

$$\begin{aligned} \sigma(x)\omega'(x) \, [\omega(x)]^{q-1} = \tau(x) \, [\omega(x)]^q - \sigma'(x) \, [\omega(x)]^q \longrightarrow \omega'(x) \, [\omega(x)]^{q-1} \\ = \frac{1}{\sigma(x)} \left(\tau(x) - \sigma'(x)\right) [\omega(x)]^q \, , \end{aligned}$$

we can write (20)

$$\int_a^b x^t [\omega(x)]^q \, dx = \frac{x^{t+1}}{t+1} [\omega(x)]^q \Big|_a^b - \int_a^b x^t \cdot x \frac{1}{t+1} q \frac{1}{\sigma(x)} \left(\tau(x) - \sigma'(x)\right) \cdot [\omega(x)]^q \, dx$$

$$= B(t, a, b) - \int_a^b x^t \cdot \left[\frac{q}{t+1} x \frac{\tau(x) - \sigma'(x)}{\sigma(x)}\right] \cdot [\omega(x)]^q \, dx$$

where

$$B(t, a, b) = \frac{1}{t+1}\left(b^t [\omega(b)]^q - a^t [\omega(a)]^q\right).$$

So, finally we have the following equation

$$\int_a^b x^t \cdot \left[1 + \frac{q}{t+1} x \frac{\tau(x) - \sigma'(x)}{\sigma(x)}\right] \cdot [\omega(x)]^q \, dx - B(t, a, b) = 0. \tag{22}$$

Remark that, since $\tau(x)$ and $\sigma(x)$ are polynomials of degrees 1 and 2 at most, the expression (22) is actually a recurrence relation of the generalized moments $\Omega_q(t)$. Thus, (19) and (22) allow us to determine analytically the weighted L_q-norm of the hypergeometric polynomials $y_n(x)$ in terms of their expansion coefficients c_k and their differential-equation coefficients $\tau(x)$ and $\sigma(x)$. To illustrate this procedure we apply it to the three classical families of orthogonal hypergeometric polynomials which satisfy the orthogonality condition (14). In these cases, it is fulfilled that $B(t, a, b) = 0$, so that (22) simplifies as

$$\int_a^b x^t \cdot \left[1 + \frac{q}{t+1} x \frac{\tau(x) - \sigma'(x)}{\sigma(x)}\right] \cdot [\omega(x)]^q \, dx = 0 \tag{23}$$

Then, taking into account the data of Table 1, this expression provides us with the following recurrence relation for the generalized moments $\Omega_q(t)$ of the Hermite, Laguerre and Jacobi polynomials:

$$\Omega_q^{(H)}(t) - \frac{2q}{t+1}\Omega_q^{(H)}(t+2) = 0 \tag{24}$$

$$\left(1 + \frac{\alpha q}{t+1}\right)\Omega_q^{(L)}(t) - \frac{q}{t+1}\Omega_q^{(L)}(t+1) = 0 \tag{25}$$

$$(t+1)\Omega_q^{(P)}(t) - q(\alpha - \beta)\Omega_q^{(P)}(t+1) - [t + 3 + q(\alpha + \beta)]\Omega_q^{(P)}(t+2) = 0, \tag{26}$$

respectively. Expression (26) is obtained from (23) by expanding the rational function that appears in the integrand in two simple power series. The resulting recurrence relation with an infinite number of terms is substracted from the same recurrence relation shifted by one unit of t to remove one of the infinite power series. This process is repeated once more to eliminate the second power series in order to finally obtain the three-term recurrence relation. Let us underline that expressions (25) and (26) hold for $\alpha \geq 0$ and $\alpha, \beta \geq 0$ in the Laguerre and Jacobi cases, respectively, because the $\Omega_q(t)$-functional (20) is well defined for arbitrary values of q only for these ranges of the parameters. Remark that the expressions (24) and (25) are two-term recurrence relation which can be solved, giving rise to the values

$$\Omega_q^{(H)}(2t+1) = 0, \quad \Omega_q^{(H)}(2t) = q^{-t-\frac{1}{2}}\Gamma\left(t+\frac{1}{2}\right) \tag{27}$$

$$\Omega_q^{(L)}(t) = q^{-(1+\alpha q+t)}\Gamma(1+\alpha q+t) \tag{28}$$

for the Hermite and Laguerre cases, respectively. As well, these values easily follow from (20). In the Jacobi case, for the reasons mentioned in Sect. 2, we have used expression (20) instead of the three-term recurrence relation (26) to obtain the value

$$\begin{aligned}\Omega_q^{(P)}(t) = \Gamma(1+t)\Big[(-1)^t \frac{\Gamma(1+\beta q)}{\Gamma(2+t+\beta q)}{}_2F_1(-\alpha q, t+1; 2+t\beta q; -1) \\ + \frac{\Gamma(1+\alpha q)}{\Gamma(2+t+\alpha q)}{}_2F_1(-\beta q, t+1; 2+t+\alpha q; -1)\Big]\end{aligned} \tag{29}$$

In summary, the expressions (19) and (27)–(29) provide a procedure to determine the weighted L_q norms ($q \in \mathbb{N}$) of the Hermite, Laguerre and Jacobi polynomials in terms of their expansion coefficients, which are well-known in the literature [1].

4 L_q-norms of general orthogonal polynomials: asymptotics ($q \to \infty$)

In this section we apply the Laplace method to obtain the asymptotic behaviour ($q \to \infty$) of the unweighted L_q-norms, $N_q(n)$, of the classical orthogonal polynomials, as given by (3). Since the Laplace method [22] demands the existence of a global maximum of the function $|y_n(x)|$, it is not applicable to the Hermite and Laguerre polynomials because the functions $|H_n(x)|$ and $|L_n^{(\alpha)}(x)|$ do not have such maximum in the intervals of orthogonality $(-\infty, +\infty)$ and $(0, +\infty)$, respectively. This is not the case for the Jacobi polynomials, which achieve the maximum

$$|P_n^{(\alpha,\beta)}(-1)| = \frac{(\beta+1)_n}{n!} \tag{30}$$

at $x = -1$ if $\beta \geq \alpha > -1,\ \beta \geq -\frac{1}{2}$ [31, Eq. 18.14.2], and

$$|P_n^{(\alpha,\beta)}(1)| = \frac{(\alpha+1)_n}{n!} \tag{31}$$

at $x = 1$ if $\alpha \geq \beta > -1,\ \alpha \geq -\frac{1}{2}$ [31, Eq. 18.14.1].

The use of an extended Laplace formula given by Theorem 1 [22, Chapter 2] allows us to obtain the following result:

Theorem 1 *The unweighted L_q norm for the Jacobi polynomials, i.e.*

$$N_q(n) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \left| P_n^{(\alpha,\beta)}(x) \right|^q dx$$

has the asymptotic behaviour

$$N_q(n) = \left(\frac{(\beta+1)_n}{n!} \right)^q \left(2^{\alpha}\Gamma(\beta+1) \left(\frac{2(\beta+1)}{(n+\alpha+\beta+1)n} \right)^{\beta+1} q^{-\beta-1} + O\left(q^{-\beta-2}\right) \right), \tag{32}$$

if $\beta \geq \alpha > -1,\ \beta \geq -\frac{1}{2}$, and

$$N_q(n) = \left(\frac{(\alpha+1)_n}{n!} \right)^q \left(2^{\beta}\Gamma(\alpha+1) \left(\frac{2(\alpha+1)}{(n+\alpha+\beta+1)n} \right)^{\alpha+1} q^{-\alpha-1} + O\left(q^{-\alpha-2}\right) \right), \tag{33}$$

if $\alpha \geq \beta > -1,\ \alpha \geq -\frac{1}{2}$.

Proof We make use of Theorem 1 [22, Chapter 2], where the asymptotic behaviour of the integral

$$I(q) = \int_a^b \phi(x) e^{-qh(x)} dx$$

is considered for $q \to \infty$, where $h(x) > h(a)\, \forall x \in (a, b)$ and the expansions

$$h(x) = h(a) + \sum_{s=0}^{\infty} a_s (x-a)^{\mu+s}$$

and

$$\phi(x) = \sum_{s=0}^{\infty} b_s (x-a)^{\gamma-1+s}$$

hold. Then, the first order of the asymptotic behaviour is given as

$$I(q) = e^{-qh(a)} \left(\Gamma\left(\frac{\gamma}{\mu}\right) \frac{b_0}{\mu a_0^{\gamma/\mu}} q^{-\frac{\gamma}{\mu}} + O\left(q^{-\frac{1+\gamma}{\mu}}\right) \right). \tag{34}$$

In our case we have that

$$\phi(x) = (1-x)^{\alpha}(1+x)^{\beta},$$

and

$$h(x) = -\ln\left|P_n^{(\alpha,\beta)}(x)\right|.$$

Let us consider first the case when $\beta \geq \alpha > -1,\ \beta \geq -\frac{1}{2}$, so that, according to Eq. (30), the maximum occurs at $x = a = -1$, fulfilling the requirement of the Theorem in [22]. Then we obtain the expansions

$$\phi(x) = 2^{\alpha}(x+1)^{\beta} + \cdots$$

so that $b_0 = 2^{\alpha},\ \gamma = \beta + 1$, and

$$h(x) = -\ln\left|P_n^{(\alpha,\beta)}(-1)\right| - \frac{1}{2}(n+\alpha+\beta+1)\frac{P_{n-1}^{(\alpha+1,\beta+1)}(-1)}{P_n^{(\alpha,\beta)}(-1)}(x+1) + \cdots,$$

so that $\mu = 1$ and

$$a_0 = -\frac{1}{2}(n+\alpha+\beta+1)\frac{P_{n-1}^{(\alpha+1,\beta+1)}(-1)}{P_n^{(\alpha,\beta)}(-1)} = \frac{1}{2}(n+\alpha+\beta+1)\frac{n}{\beta+1}.$$

The substitution of these values of $a_0,\ b_0,\ \gamma$ and μ in Eq. (34) gives rise to the desired result (32) of the Theorem for $\beta \geq \alpha > -1,\ \beta \geq -\frac{1}{2}$. Similarly, with the change of variable $x \to -x$, the result (33) for $\alpha \geq \beta > -1,\ \alpha \geq -\frac{1}{2}$, is obtained. □

5 Entropy-like integrals of Bessel polynomials

In this section we determine the following parametric families of entropy-like integrals

$$\tilde{N}_q(n) \equiv \frac{1}{2\pi i}\int\limits_{\Delta} \omega(x)\,(y_n(x))^q\ dx,\ \ q = 2k, k \in \mathbb{N}, \tag{35}$$

and

$$\tilde{W}_q(n) \equiv \frac{1}{2\pi i}\int\limits_{\Delta} \left(\omega(x)y_n^2(x)\right)^q\ dx,\ \ q \in \mathbb{N}, \tag{36}$$

for the Bessel hypergeometric polynomials $\mathcal{B}_n^{(\alpha)}(z),\ \alpha = 1, 2, \ldots$, [24–27,31]. Since the Bessel polynomials take complex values, these integrals have not the probabilistic interpretation of unweighted [Eq. (3)] and weighted [Eq. (4)] L_q norms of these polynomials, respectively. Nevertheless, they can be evaluated by means of expansions (6) and (19), respectively.

To evaluate $\tilde{N}_q(n)$, we follow the procedure developed in Sect. 2 to reach the recurrence relation (13). The condition $A(t, a, b) = 0$ is also fulfilled by the Bessel polynomials because they are orthogonal in the unit circle of the complex plain so that $a = b$, and the weight function is not multivalued. Finally we obtain that

$$\tilde{N}_q(n) = \sum_{t=0}^{nq} \frac{q!}{(t+q)!} B_{t+q,q}(c_0, 2!c_1, \ldots, (t+1)!c_t)\mu_t(\mathcal{B}), \tag{37}$$

where c_i are the coefficients of the expansion of the polynomial in terms of ordinary powers [24–26], and μ_t denotes the Krall–Frink moment [26] of order t of the weight function $\omega(x)$; i.e.

$$\mu_t(\mathcal{B}) = \frac{1}{2\pi i} \int\limits_{\Delta} x^t \omega(x)\, dx; \, t = 0, 1, \ldots \tag{38}$$

These moments satisfy the two-term recurrence relation

$$(t + \alpha + 2)\mu_{t+1}(\mathcal{B}) + 2\mu_t(\mathcal{B}) = 0,$$

where we have operated as done in Sect. 2 and we have taken into account the main data of Bessel polynomials given in Table 1. This recurrence relation can be exactly solved, giving rise to the following explicit expression:

$$\mu_t(\mathcal{B}) = \frac{(-2)^{\alpha+t+1}}{\Gamma(\alpha+t+2)}.$$

For the entropy-like integrals $\tilde{W}_q(n)$, defined in Eq. (36), we now follow an analogous procedure to that shown in Sect. 3, yielding the expression

$$\tilde{W}_q(n) = \sum_{t=0}^{2nq} \frac{(2q)!}{(t+2q)!} B_{t+2q,2q}(c_0, 2!c_1, \ldots, (t+1)!c_t) \cdot \Omega_q^{(\mathcal{B})}(t),$$

where

$$\Omega_q^{(\mathcal{B})}(t) = \frac{1}{2\pi i} \int\limits_{\Delta} x^t [\omega(x)]^q \, dx.$$

These generalized moments $\Omega_q(t)$ satisfy the recurrence relation

$$\left(1 + \frac{\alpha q}{t+1}\right) \Omega_q^{(\mathcal{B})}(t) + \frac{2q}{t+1} \Omega_q^{(\mathcal{B})}(t-1) = 0,$$

that yields the following expression for the Bessel polynomials.

$$\Omega_q^{(\mathcal{B})}(t) = (-1)^{\alpha q+t+1} \frac{2^{\alpha q+t+1} q^{\alpha q+t+1}}{\Gamma(\alpha q + t + 2)}. \tag{39}$$

In doing so, we have operated as in Sect. 3 and we have taken into account the main data of Bessel polynomials given in Table 1.

Finally, let us point out that the calculation of the leading term of the asymptotics of the integrals $\tilde{N}_q(n)$ and $\tilde{W}_q(n)$ for $q \to \infty$ of Bessel polynomials requires the use of an approach of the steepest descents type, what lies beyond the scope of this paper.

6 Conclusions and open problems

The L_q-norms of probability distributions play an important role in various mathematical fields ranging from harmonic analysis to approximation theory. Moreover, the L_q-norms of the Rakhmanov probability density associated to the orthogonal hypergeometric polynomials $y_n(x)$ describe some entropic or information-theoretic quantities of these functions which measure their spreading over their orthogonality interval in different manners. In this work we have given a procedure to determine the unweighted and weighted L_q-norms (q positive integer) of the general hypergeometric polynomials in terms of q and the parameters of the corresponding weight function. It is mainly based on the expansion (6) of the powers of the involved polynomials $y_n(x)$, whose coefficients are multivariate Bell polynomials with variables given by the coefficients c_k of the explicit expression (5) of the polynomials $y_n(x)$, which are well known in the literature (see e.g. [2,31]). Then we have applied this procedure to the classical real orthogonal polynomials (Hermite, Laguerre and Jacobi), so that the L_q-norms are expressed in terms of the known expansion coefficients of the polynomials, whose values are well-known in the literature (see e.g. [1,2,31]).

In the case of the Bessel polynomials we have used the same techniques employed previously for the real hypergeometric polynomials to obtain some entropy-like integrals, formally related to the weighted and unweighted L_q-norms, but they do not possess a probabilistic interpretation due to the complex character of the polynomials.

In addition, since the resulting expression may be highbrow for large and very large values of q, we have used the extended Laplace method [22] to tackle the asymptotics ($q \to \infty$) of the unweighted L_q-norms of the classical orthogonal polynomials of Jacobi type. This method does not work in the Hermite and Laguerre cases, as already discussed. Moreover, in the Bessel case we need to use an approach of the steepest descents type, what is beyond the scope of this work.

Let us underline that since the orthogonal hypergeometric polynomials control the physical solutions of the Schrödinger equation of numerous quantum-mechanical potentials, this work contributes to pave the way to determine the entropic moments and the entropies of Rényi and Tsallis types of a great deal of physical and chemical systems in an analytical manner.

Finally, in addition to find some relevant properties for the L_q-norms from the closed expressions obtained in this work, we are aware that it would be desirable (almost mandatory, because of its mathematical, physical and chemical interest) to find the L_q-norms of the hypergeometric orthogonal polynomials for any real positive number q. However this would require a completely different approach than the one presented in this work, which is still unknown in the literature. The only alternative method [32] existing by now is also valid for integer positive values of q; it is based on some extended linearization formulas of $y_n(x)$, which allow us to express the L_q-norms in terms of some generalized Lauricella and Srivastava-Daoust functions with parameters and variables given by $n,\ q$ and the parameters of the weight function of the involved polynomials.

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